Q.1

Show that if a series is conditionally convergent, then the series obtained from its positive terms is divergent, and the series obtained from its negative terms is divergent.

Solution:

Recall that $\sum a_n$ is conditionally conv.

$\Leftrightarrow$ $\sum a_n$ is conv. but $\sum |a_n|$ is divergent.

Note that

$$p_n = \frac{a_n + |a_n|}{2} = \begin{cases} a_n & \text{if } a_n \geq 0 \\ 0 & \text{if } a_n < 0 \end{cases}$$

are all the non-negative terms in (a_n), and

$$q_n = \frac{a_n - |a_n|}{2} = \begin{cases} a_n & \text{if } a_n \leq 0 \\ 0 & \text{if } a_n > 0 \end{cases}$$

are all the non-positive terms in (a_n).

We do a proof by contradiction. Suppose at least one of $\sum p_n$ or $\sum q_n$ is conv.

Case 1: $\sum p_n$ is conv.

Then $\sum |a_n| = \sum (2p_n - a_n)$, which is conv.

This contradicts the cond. conv. of $\sum a_n$.

<u>Case 2</u>: $\sum q_n$ is conv.

Then $\sum |a_n| = \sum (a_n - 2q_n)$, which is conv.

This contradicts the cond. conv. of $\sum a_n$.

Q.2

Show that $\frac{1}{1^2}+\frac{1}{2^3}+\frac{1}{3^2}+\frac{1}{4^3}+\cdots$ is convergent, but that both the Ratio and the Root Test fails to apply.

Solution:

$$\frac{1}{1^2}+\frac{1}{2^3}+\frac{1}{3^2}+\frac{1}{4^3}+\cdots = \sum a_n$$

where $a_n = \begin{cases} \frac{1}{n^2} & \text{if } n=2k-1 \\ \frac{1}{n^3} & \text{if } n=2k. \end{cases}$

Ratio test fails:

Note that

$$\frac{a_{n+1}}{a_n} = \begin{cases} \frac{n^2}{(n+1)^3} & , n=2k-1 \\ \frac{n^3}{(n+1)^2} & , n=2k \end{cases}$$

$$\therefore \lim_{k\to\infty} \frac{a_{2k}}{a_{2k-1}} = \lim_{k\to\infty} \frac{(2k-1)^2}{(2k)^3} = 0$$

$$\lim_{k\to\infty} \frac{a_{2k+1}}{a_{2k}} = \lim_{k\to\infty} \frac{(2k)^3}{(2k+1)^2} = \infty$$

$\therefore \nexists\ K \in \mathbb{N}$ and $r \in (0,1)$ s.t. $\left|\frac{x_{n+1}}{x_n}\right| \leq r \quad \forall n \geq K$

or $\left|\frac{x_{n+1}}{x_n}\right| \geq 1 \quad \forall n \geq K.$

$\therefore$ Ratio test fails.

Root test fails:

$$|a_n|^{1/n} = \begin{cases} \dfrac{1}{n^{2/n}} & \text{if } n = 2k-1 \\ \dfrac{1}{n^{3/n}} & \text{if } n = 2k. \end{cases}$$

Note that for each $p > 0$,

$$\lim_{x\to\infty} x^{-p/x} = \lim_{x\to\infty} e^{-p\frac{\ln x}{x}}$$

$$= \lim_{x\to\infty} e^{-p \cdot \frac{1}{x}} \qquad \text{(L'Hôpital's)}$$

$$= 1$$

$$x^{-p/x} < 1 \iff -\frac{p}{x}\ln x < 0 \iff x > 1$$

$\therefore$ $|a_n|^{\frac{1}{n}} < 1$ and $|a_n|^{\frac{1}{n}} \to 1$ as $n \to \infty$.

If $\exists K \in \mathbb{N}$ and $r < 1$ s.t. $|a_n|^{1/n} \le r$ for $n \ge K$, then

$$\lim_{n\to\infty} |a_n|^{1/n} \le r < 1$$

which is a contradiction.

$\therefore$ Root test fails.

$\frac{1}{1^2} + \frac{1}{2^3} + \frac{1}{3^2} + \frac{1}{4^3} + \cdots$ converges:

We do a limit comparison test with $(b_n) = (\frac{1}{n^{3/2}})$

$$\left|\frac{a_n}{b_n}\right| = \begin{cases} n^{-1/2} & \text{for } n=2k-1 \\ n^{-3/2} & \text{for } n=2k \end{cases} \longrightarrow 0 \quad \text{as } n \to \infty$$

Since $\sum b_n$ is abs. conv., $\sum a_n$ is abs. conv. by the limit comparison test.

Q.3

Let $0<a<1$ and consider the series

$$a^2+a+a^4+a^3+\dots+a^{2n}+a^{2n-1}+\dots$$

Show that the root test applies, but that the ratio test does not apply.

Solution:

Root test:

$$a^2+a+a^4+a^3+\dots+a^{2n}+a^{2n-1}+\dots = \sum b_n$$

where

$$b_n = \begin{cases} a^{n+1} & \text{for } n=2k-1 \\ a^{n-1} & \text{for } n=2k \end{cases}$$

$$\therefore |b_n|^{1/n} = \begin{cases} a^{1+\frac{1}{n}} & \text{for } n=2k-1 \\ a^{1-\frac{1}{n}} & \text{for } n=2k \end{cases} \longrightarrow a \quad \text{as } n\to\infty$$

Since $0<a<1$, $\sum b_n$ is conv. by the root test.

Ratio test:

$$\left|\frac{b_{n+1}}{b_n}\right| = \begin{cases} a^{-1} > 1 & \text{, for } n = 2k-1 \\ a^3 < 1 & \text{, for } n = 2k \end{cases}$$

$\therefore \nexists K \in \mathbb{N}$ and $r \in (0,1)$ s.t. $\left|\frac{b_{n+1}}{b_n}\right| \leq r \quad \forall n \geq K$

or $\left|\frac{b_{n+1}}{b_n}\right| \geq 1$

$\therefore$ Ratio test does not apply.